Based on K. H. Rosen: Discrete Mathematics and its Applications.

Lecture 7: Sets, power sets, Cartesian products. Section 2.1

1 Sets, power sets, Cartesian products.

Sets are of great importance in mathematics; in modern formal treatments, most mathematical objects (numbers, relations, functions, etc.) are defined in terms of sets.

Definition 1. A **set** is an unordered collection of objects, called elements or members of the set. A set is said to contain its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

Definition 2. Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if

$$\forall x \, (x \in A \iff x \in B)$$

We write $A = B$ if A and B are equal sets.

The classical numerical domains are denoted as follow:

1. The natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.
2. The integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
3. The rationals $\mathbb{Q} = \{\frac{p}{q} \,|\, p, q \in \mathbb{Z} \text{ and } q \neq 0\}$.
4. The real numbers $\mathbb{R}$.
5. The complex numbers $\mathbb{C}$.

There is a special set that has no elements. This set is called the empty set, or null set, and is denoted by $\emptyset$. For example

$$\emptyset = \{x \in \mathbb{R} \,|\, x > 0 \text{ and } x > 2x\}$$

or

$$\emptyset = \{x \in \mathbb{Q} \,|\, x^2 = 2\}.$$

Definition 3. The set A is **a subset** of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

Example 4. For every subset S we have: $\emptyset \subseteq S$ and $S \subseteq S$.

Example 5. Consider the set $A = \{\{a\}, \{b\}, c\}$ and the set $\{c\}$ made of one element. The set $\{c\}$, is a subset of A. On the other hand the set $\{a\}$ is a member or an element of A. We write

$$\{c\} \subseteq A \qquad \text{and} \qquad \{a\} \in A.$$

Remark 6. To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Definition 7. Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the **cardinality** of S. The cardinality of S is denoted by $|S|$. A set is said to be infinite if it is not finite.

Definition 8. Given a set S, **the power set** of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

Definition 9. The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, . . . , and a_n as its nth element. In particular, ordered 2-tuples are called ordered pairs.

Definition 10. Let A and B be sets. The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all **ordered pairs** (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

Remark 11. A subset R of the Cartesian product $A \times B$ is called a relation from the set A to the set B.

Questions

(a) List all subsets of the 3-element set $A = \{1, 2, 3\}$. How many subsets are there?